

ON THE HILBERT SCHEME OF THE MODULI SPACE OF VECTOR BUNDLES OVER AN ALGEBRAIC CURVE

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ABSTRACT. Let $M(n, \xi)$ be the moduli space of stable vector bundles of rank $n \geq 3$ and fixed determinant ξ over a smooth projective algebraic curve X over \mathbb{C} of genus $g \geq 4$. We use the gonality of the curve and r -Hecke morphisms to describe a smooth open set and to compute the dimension of a component of the Hilbert scheme $\text{Hilb}_{M(n, \xi)}$, of the scheme of morphisms $\text{Mor}(\mathbb{G}, M(n, \xi))$ and of the moduli space $M_{X \times \mathbb{G}}$ of stable bundles over $X \times \mathbb{G}$, where \mathbb{G} is the Grassmannian $\mathbb{G}(n-r, \mathbb{C}^n)$. In particular, we prove that $\dim \text{Mor}_P(\mathbb{P}^2, M(3, \xi)) = 8g - 7$ and we give a sufficient condition for $\text{Mor}_{2ns}(\mathbb{P}^1, M(n, \xi))$ to be non-empty with $s \geq 1$.

1. INTRODUCTION

Studying the geometry of varieties, frequently involves the understanding of their subvarieties. Roughly speaking, the Hilbert schemes Hilb_Y , the Chow schemes $C(Y)$ and the $\text{Mor}(-, Y)$ schemes are schemes that parameterise subvarieties of a projective variety Y . It is clear that a subscheme $Z \subset Y$ defines an element in Hilb_Y , in $\text{Mor}(Z, Y)$ and in $C(Y)$; and, in general, there is a subset that is common to all three schemes. The Hilbert scheme, in general, is different from the $\text{Mor}(-, Y)$ scheme, since the limit of a subscheme could be different from the limit of the associated map. Recall that the Hilbert scheme compactifies embedded subvarieties by allowing them to degenerate into arbitrary subschemes.

In this paper, we are interested in studying the Hilbert scheme Hilb_M^P of the moduli space $M(n, \xi)$ of stable vector bundles of rank $n \geq 3$ and fixed determinant ξ over a smooth projective algebraic curve X of genus $g \geq 4$, for a fixed Hilbert polynomial P . The Hilbert schemes, in general, can have a very complicated structure; they can be non connected or could even have a component that is non-reduced everywhere.

For the moduli space $M(2, \mathcal{O})$, it was proved in [NR2] that there exists a component of the Hilbert scheme of $M(2, \mathcal{O})$ that provides a nonsingular model for $M(2, \mathcal{O})$. Different compactifications of $M(2, \mathcal{O})$ were given in [CCK]. For

2000 *Mathematics Subject Classification.* 14H60, 14J60.

Key words and phrases. moduli spaces, Hecke cycles, Grassmannian, Hilbert scheme, morphisms.

$M = M(2, \mathcal{O}_X(-1))$ the Hilbert scheme $Hilb_M^{m+1}$ and the Chow scheme of degree 1 curves, with respect to the ample generator Θ , coincide with the moduli space of stable maps $Mor_1(\mathbb{P}^1, M)$ (see [Mu] and [Kil]). In [Ki] Kiem proved that the Hilbert scheme $Hilb_M^{2m+1}$ of conics, the Chow scheme of conics and $Mor_2(\mathbb{P}^1, M)$ are related by contractions. To prove these results, they used the space of Hecke curves. Recall that the *Hecke correspondence* for vector bundles, which has been one of the most powerful tools in the study of $M(n, \xi)$, was described by Narasimhan and Ramanan in [NR1] and refers to the following diagram

$$\begin{array}{ccc} & \mathfrak{h} & \\ \pi \swarrow & & \searrow q \\ M(n, \nu) & & M(n, \xi), \end{array}$$

with π and q projective fibrations over suitable open sets. Tyurin in [T] made a similar construction and obtained Grassmannian fibrations. They used $(0, 1)$ and $(1, 0)$ -stable bundles (see §2.3) to define subschemes of $M(n, \xi)$, called *good Hecke cycles*, that are isomorphic to projective spaces. The Hecke cycles were the main concern in [NR2]. However, they also introduce the *Hecke curves*, that are the lines in the Hecke cycles, and their properties were implicitly studied there. Actually, they turn out to be the minimal rational curves in $M(n, \xi)$ (see [H2], [MS] and [S]). Their corresponding tangent spaces and the moduli space of rational curves have been studied principally for rank 2 in [MS], [H1], [H2], [CCK], [S] and [C]. The interest in studying $Mor(\mathbb{P}^1, M(n, \xi))$ also draws its origin from attempts to compute the quantum cohomology of the moduli space $M(n, \xi)$, which has recently become an important topic of research (see e.g. [Mu]; [W]; [BDW]; [B] and [Ki]).

The space that parametrises the lines in a projective space $\mathbb{P}(V)$ is the Grassmannian $\mathbb{G}(1, \mathbb{P}(V))$ which is isomorphic to $\mathbb{G}(2, V)$. The problem that we address in this article is the description of the space that parameterises Grassmannian $\mathbb{G} = \mathbb{G}(n - r, \mathbb{C}^n)$ in $M(n, \xi)$, for any $r \geq 1$. Our purpose is to describe a smooth open set of an irreducible component of $Hilb_M^P$ and of $Mor(\mathbb{G}, M(n, \xi))$. It will be interesting to prove, in the non-coprime case, that our component is actually smooth and provides a non-singular model for $M(n, \xi)$, as was proved for $n = 2$ and ξ the trivial bundle in [NR1].

In order to state our results, we recall that O. Mata-Gutiérrez in [M2] (see also [T]), using (k, ℓ) -stable bundles, defines the *Hecke Grassmannians* in $M(n, \xi)$, that generalizes the idea of the good Hecke cycles. He proves that through a very general point $E \in M(n, \xi)$, there exists Hecke Grassmannians passing through E and describes the scheme of such Grassmannians. The Hecke Grassmannian is called *r-Hecke cycle* if it defines a closed subscheme. The good Hecke cycles are precisely those where $r = 1$.

The set of r -Hecke cycles form an irreducible family (see §3). Let \mathcal{HG} be the irreducible component of the Hilbert scheme Hilb_M^P of $M(n, \xi)$ containing r -Hecke cycles where

$$P(m) = \chi(\mathbb{G}, m\mathcal{O}_{\mathbb{G}}(2n))$$

is the Hilbert polynomial associated to the r -Hecke cycles that were defined in [M2]. The main idea behind the study of \mathcal{HG} is to apply the (k, ℓ) -stability. We construct a fibration \mathcal{A} over X with fibre at $x \in X$ the set of (k, ℓ) -stable vector bundles of rank n and determinant $\xi(rx)$, where k, ℓ and $1 \leq r \leq n-1$ are integers so chosen that

$$0 \leq k(n-1) + \ell + r < (n-1)(g-1)$$

and

$$0 \leq k + (\ell + r)(n-1) < (n-1)(g-1).$$

To prove the next theorem, we assume that universal bundles exist. However, in the non-coprime case, we can always use an étale cover M' and a universal bundle over $X \times M'$ and make a similar construction (see [NR1, Proposition 2.4]).

Theorem 1.1. *If $(n, d) = 1$ and r is less than the gonality of X then*

- (1) *there is an algebraic isomorphism Υ from \mathcal{A} to an open subscheme of \mathcal{HG} .*
- (2) *The Hilbert scheme is smooth at $[\Upsilon(z)]$, for any $z \in \mathcal{A}$.*
- (3) $\dim \mathcal{HG} = (n^2 - 1)(g - 1) + 1$.
- (4) *The deformations of r -Hecke cycles are r -Hecke cycles.*

The proof is similar in spirit to the proof of [NR2, Theorem 5.13] (see also [T]). The difference is on the target and the open set that is considered. In this paper, the target is $M(n, \xi)$. Some of the results in [NR1] and [NR2] apply directly to our case, others need an extra hypothesis, like that of the gonality of the curve, and others are completely different to our case. We prove the corresponding results that we need. In our case we have a diagram (see diagram 3.9)

$$\begin{array}{ccc} & \mathbb{G} & \\ \pi_1 \swarrow & & \searrow \Phi \\ \mathcal{A} & & M(n, \xi) \end{array}$$

with π_1 a Grassmannian fibration and, for each F in a suitable open set $\mathcal{B} \subset M(n, \xi)$, the fibre of Φ at F is the Grassmannian bundle $p : \mathbb{G}(r, F) \rightarrow X$.

We use the gonality of the curve to prove that any $z \in \mathcal{A}$ defines a closed embedding $\phi_z : \mathbb{G} \rightarrow M(n, \xi)$ of a Grassmannian \mathbb{G} to $M(n, \xi)$, and for the injectivity of $\Upsilon : \mathcal{A} \rightarrow \mathcal{HG}$. Then, to compute the differential $d\Upsilon$, we prove (see Proposition 4.2) that $T_z\mathcal{A} = H^0(\mathbb{G}, N_{\mathbb{G}/M})$ and $H^i(\mathbb{G}, N_{\mathbb{G}/M}) = 0$ for $i \geq 1$, where $N_{\mathbb{G}/M}$ is the normal bundle of $\phi_z(\mathbb{G})$ in $M(n, \xi)$. Moreover, $N_{\mathbb{G}/M}$

is generated by global sections and small deformations of r -Hecke cycles are again r -Hecke cycles.

One of the advantages of using Grassmannians \mathbb{G} lies in the fact that they have natural subvarieties, namely the Schubert varieties, and this can allow us to study flag Hilbert schemes parameterising t -tuples $(\mathbb{G}, Y_1, \dots, Y_t)$ such that $Y_1 \subset \dots \subset Y_t \subset \mathbb{G}$ are subschemes of $M(n, \xi)$ with Hilbert polynomials P_0, \dots, P_t , respectively. However, this topic exceeds the scope of this paper. We just consider nested pairs of subschemes $\mathbb{P}^1 \xrightarrow{i} \mathbb{G} \xrightarrow{j} M(n, \xi)$. In particular, our viewpoint sheds some new light on the study of rational curves on $M(n, \xi)$ allowing us to obtain many more rational curves of higher degrees.

Corollary 1.2. *If $Mor_s(\mathbb{P}^1, \mathbb{G}) \neq \emptyset$ then $Mor_{2ns}(\mathbb{P}^1, M(n, \xi)) \neq \emptyset$. Moreover, $\dim Mor_{2ns}(\mathbb{P}^1, M(n, \xi)) \geq (n^2 - 1)(g - 1) + 1$.*

If we start with at least $(1, r)$ -stable bundles then the rational curves in Corollary 1.2 will pass through a general point. However, if we consider $(0, r)$ -stable bundles, the rational curves could be in a closed set (see [M2]).

The fibration \mathcal{A} parameterises closed subschemes, morphisms $f : \mathbb{G} \rightarrow M(n, \xi)$ as well as stable vector bundles over $X \times \mathbb{G}$. To see this, let us say that a morphism $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ is an r -Hecke morphism if $\lambda(\mathbb{G})$ is an r -Hecke cycle. Such morphisms have degree $2n$ (see Proposition 4.6) and are in an irreducible family parameterised by \mathcal{A} . Let P be the Hilbert polynomial $P(m) = \chi(\mathbb{G}, m(\mathcal{O}_{\mathbb{G}}(2n)))$, and $Hom_P^H(\mathbb{G}, M(n, \xi))$ the irreducible component of the scheme $Hom_P(\mathbb{G}, M(n, \xi))$ that contains the r -Hecke morphisms. We denote by $Mor_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi))$ the quotient

$$Hom_P^H(\mathbb{G}, M(n, \xi)) / Aut(\mathbb{G}).$$

The next theorem (see Proposition 5.3, Theorem 5.4 and Corollary 5.6) is a consequence of Theorem 1.1.

Theorem 1.3. *If $(n, d) = 1$ and r is less than the gonality of X then there is an algebraic injective morphism*

$$\Sigma : \mathcal{A} \rightarrow Mor_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi)).$$

Moreover, $Mor_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi))$ is smooth at the r -Hecke morphisms and

$$\dim Mor_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi)) = (n^2 - 1)(g - 1) + 1.$$

In particular, $\dim Mor_P^{\mathcal{H}, 1}(\mathbb{P}^2, M(3, \xi)) = 8g - 7$.

For any $z = (x, E) \in \mathcal{A}$, the Hecke morphism ϕ_z is defined by the existence of a vector bundle \mathcal{P}_z over $X \times \mathbb{G}$, which we shall call an r -Hecke bundle. The stability of \mathcal{P}_z , with respect to any polarisation L , is established in Proposition 5.7. Let $M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$ be the irreducible component of the moduli space of

L -stable vector bundles over $X \times \mathbb{G}$ that contains \mathcal{P}_z . Our last theorem (see Theorem 5.8) describes a smooth open set of $M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$ and we see that, locally, the deformations of \mathcal{P}_z come from those of the curve, of the Grassmannian as well as those induced by the elements of $H^0(\mathbb{G}, \phi_z^*(TM))$.

Theorem 1.4. *If $(n, d) = 1$ and r is less than the gonality of X then*

- (1) *there is an algebraic injective morphism $\Gamma : \mathcal{A} \rightarrow M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$.*
- (2) *$M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$ is smooth at $\Gamma(z)$, for all $z \in \mathcal{A}$.*
- (3) *$\dim M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z) = n^2g + 1$.*
- (4) *$\dim M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)/\text{Aut}(X \times \mathbb{G}) = (n^2 - 1)(g - 1) + 1$.*

This article is organised as follows. In section §2, we summarize the relevant material on Grassmannians, (k, ℓ) -stability and elementary transformations. Some of the recent results are reviewed in a more general setting. Section 3 is devoted to the study of the morphisms ϕ_z and Υ . The main result is proved in the fourth section. In §5, we touch only a few aspects of the theory of rational curves on $M(n, \xi)$ and prove Theorem 1.3 and 1.4, and we raise some questions in section 6.

Notation: Given a vector bundle E over X we denote by d_E the degree, by n_E the rank and by $\mu(E) := \frac{d_E}{n_E}$ the slope of E . The Grassmannian of s -planes of a vector space or of a vector bundle V will be denoted by $\mathbb{G}(s, V)$. By $\mathcal{O}_Y \times W$, we mean the trivial bundle over Y with fibre the vector space W . By p_i , we mean the natural projection in the i -factor.

Acknowledgments: We would like to thank Tomás Gómez for his contributions to our understanding of the Hecke correspondence. We thank the Isaac Newton Institute, Cambridge, where this work start, for its very stimulating mathematical environment, and to CONACYT Grant-128250 for the support. The first author is member of the research group VBAC and thanks Jochen Heinloth, R. Ramadas and Ugo Bruzzo for very helpful conversations as well as Vicente Muñoz and V. Balaji for their comments and suggestions about the paper. She also thanks CMR, Barcelona and ICTP, Trieste for their hospitality as well as for the support during completion of this work, and for the great research environment.

2. GRASSMANNIANS, (k, ℓ) -STABILITY AND ELEMENTARY TRANSFORMATIONS.

Let X be a smooth projective algebraic curve over \mathbb{C} of genus $g \geq 4$. Let $M(n, d)$ be the moduli space of stable vector bundles of rank $n \geq 3$ and degree d over X and $M(n, \xi)$ the moduli space of stable vector bundles of rank n with fixed determinant ξ of degree d . It is well known that $M(n, \xi)$ is a Fano variety with Picard group $\mathbb{Z}\Theta$, where Θ is an ample divisor. If K_M is the canonical

bundle then $-K_M = 2(n, d)\Theta$ (see [DR]). When n and d are coprime $M(n, \xi)$ is projective, smooth of dimension $(n^2 - 1)(g - 1)$ and there is a Poincaré bundle \mathcal{U} and a Grassmannian Poincaré bundle $\mathbb{G}(s, \mathcal{U})$. If n and d have a common divisor there is no universal vector bundle (see [R1]; also [N]). In [NR1, Proposition 2.4] it was proved that in the non-coprime case there exist an étale cover of $M(n, \xi)$ and a family \mathcal{V} of stable bundles of rank n and determinant ξ with universal properties. However, as in the projective case (see [BBN]), there always exist a Grassmannian Poincaré bundle

$$\mathbb{G}(s, \mathcal{U}) \rightarrow X \times M(n, \xi)$$

with the property that its restriction to $X \times \{E\}$ is isomorphic to the Grassmannian bundle $\mathbb{G}(s, E)$ over X that parameterises all the s -planes in the fibres of the stable vector bundle E on X .

Since Grassmannians, elementary transformations and (k, ℓ) -stability will be the main tools that we will use, we briefly review the essentials.

2.1. Grassmannians. Most of this subsection is a direct generalization of the corresponding results for projective spaces. We will give only a brief exposition of the principal properties of Grassmannians of vector spaces and of vector bundles that we use; and we fix the notation.

Let E be a vector bundle of rank n over X and let E_x be the fibre at $x \in X$. Let $\mathbb{G}(n - r, E_x)$ be the Grassmannian of $(n - r)$ -planes of E_x . By associating with a $n - r$ dimensional subspace of E_x the r dimensional orthogonal subspace of the dual space $(E_x)^*$, we see that $\mathbb{G}(n - r, E_x) = \mathbb{G}(r, E_x^*)$.

There is the tautological exact sequence

$$(2.1) \quad \xi_{E_x} : 0 \rightarrow \mathcal{S}_{E_x} \rightarrow \mathcal{O}_{\mathbb{G}} \times E_x \xrightarrow{\alpha_{E_x}} \mathcal{Q}_{E_x} \rightarrow 0$$

over $\mathbb{G}(n - r, E_x)$, where \mathcal{S}_{E_x} and \mathcal{Q}_{E_x} are the tautological bundles of rank $n - r$ and r , respectively. It is well known that $\det(\mathcal{Q}_{E_x}) = \mathcal{O}_{\mathbb{G}}(1)$ and $H^i(\mathbb{G}(n - r, E_x), \mathcal{S}_{E_x}) = 0$ for all $i \geq 0$.

Remark 2.1. For $\mathbb{G}(n - r, E_x)$

- (1) the tangent bundle $T\mathbb{G}$ of $\mathbb{G}(n - r, E_x)$ is $\mathcal{S}_{E_x}^* \otimes \mathcal{Q}_{E_x}$; has degree n ; and $\dim H^0(\mathbb{G}(n - r, E_x), T\mathbb{G}) = n^2 - 1$ and $H^i(\mathbb{G}(n - r, E_x), T\mathbb{G}) = 0$ for $i \geq 1$.

- (2) If $\Omega^1\mathbb{G}$ is the cotangent bundle then

$$(2.2) \quad H^i(\mathbb{G}(n - r, E_x), \Omega^1\mathbb{G}) = 0, \text{ for } i \neq 1 \text{ and } H^1(\mathbb{G}(n - r, E_x), \Omega^1\mathbb{G}) = \mathbb{C}.$$

Let $p_E : \mathbb{G}(n - r, E) \rightarrow X$ be the Grassmannian bundle whose fibre at $x \in X$ is $\mathbb{G}(n - r, E_x)$ and let

$$(2.3) \quad \xi_E : 0 \rightarrow \mathcal{S}_E \rightarrow p_E^* E \xrightarrow{\alpha_E} \mathcal{Q}_E \rightarrow 0$$

be the tautological exact sequence where S_E and Q_E are the tautological bundles.

Remark 2.2. For $p_E : \mathbb{G}(n-r, E) \rightarrow X$

(1) the surjective morphism $E_x \rightarrow E_x/W$ allow us to identify a point $\wp = (x, E, W \subset E_x)$ in $\mathbb{G}(n-r, E)$ with the pair $(x, E \xrightarrow{\rho} \mathcal{O}_x^r)$ where $\text{Ker } \rho_x = W$.

(2) The tangent bundle of $\mathbb{G}(n-r, E)$, denoted by $T\mathbb{G}(E)$, fits in the following extension

$$(2.4) \quad \zeta : 0 \rightarrow T_{p_E} \rightarrow T\mathbb{G}(E) \rightarrow p_E^*TX \rightarrow 0,$$

where T_{p_E} is the tangent in the fibres and $T_{p_E} = S_E^* \otimes Q_E$.

The Grassmannians varieties \mathbb{G} are Fano varieties with Picard number 1. Given a morphism $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ we define the degree of λ as

$$d(\lambda) := \text{degree}(\lambda^*(-K_M)).$$

From [G, 4(c)], the morphisms from \mathbb{G} to $M(n, \xi)$ are parameterised by a locally Noetherian scheme $\text{Hom}(\mathbb{G}, M(n, \xi))$. We can see $\text{Hom}(\mathbb{G}, M(n, \xi))$ as a subscheme of $\text{Hilb}_{(\mathbb{G} \times M(n, \xi))}$ when we identified a morphism λ with its graph Γ_λ in $\mathbb{G} \times M(n, \xi)$. Recall that $\text{Hom}(\mathbb{G}, M(n, \xi))$ is the disjoint union of the subschemes $\text{Hom}_P(\mathbb{G}, M(n, \xi))$, for all polynomials P , where $\text{Hom}_P(\mathbb{G}, M(n, \xi))$ is the subscheme that parameterises morphisms $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ with fixed Hilbert polynomial $P(m) = \chi(\mathbb{G}, m\lambda^*(-K_M))$. For a fixed polynomial P , denote by $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ scheme

$$\text{Hom}_P(\mathbb{G}, M(n, \xi))/\text{Aut}(\mathbb{G}),$$

where $\lambda \sim \lambda'$ if there exists $\beta \in \text{Aut}(\mathbb{G})$ such that $\lambda' = \lambda \circ \beta$.

Remark 2.3. From [G] and [Ko] we have that

- (1) the expected dimension of $\text{Hom}_P(\mathbb{G}, M(n, \xi))$ is $h^0(\mathbb{G}, \lambda^*TM(n, \xi))$.
- (2) $\text{Hom}_P(\mathbb{G}, M(n, \xi))$ is smooth at λ if $h^1(\mathbb{G}, \lambda^*TM(n, \xi)) = 0$.
- (3) Recall that $H^0(\mathbb{G}, T\mathbb{G})$ is the tangent space at the identity to the group of automorphisms of \mathbb{G} , and the image of the canonical morphism

$$H^0(\mathbb{G}, T\mathbb{G}) \rightarrow H^0(\mathbb{G}, \lambda^*(TM))$$

corresponds to the deformations of λ by reparameterisations.

- (4) The expected dimension of $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ is the dimension of

$$H^0(\mathbb{G}, \lambda^*(TM))/H^0(\mathbb{G}, T\mathbb{G}).$$

- (5) $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ is smooth at λ if $h^1(\mathbb{G}, \lambda^*TM(n, \xi)) = 0$.

To define morphisms $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ we use the *r-elementary transformations* and *(k, ℓ)-stability*.

2.2. r -Elementary transformations. Let E be a vector bundle over X of rank n and determinant η of degree e . For any $x \in X$ and $W \in \mathbb{G}(n-r, E_x)$ consider a new vector bundle E^W defined by the exact sequences of sheaves

$$(2.5) \quad \xi_{x,W} : 0 \rightarrow E^W \xrightarrow{\iota} E \xrightarrow{\alpha_W} \mathcal{O}_x \times (E_x/W) \rightarrow 0,$$

where $\mathcal{O}_x \times (E_x/W)$ is the skyscraper sheaf with support in x and fibre E_x/W . The vector bundle E^W is called the *r -elementary transformation of E in (x, W)* . Note that E^W has rank n , degree $e - r$ and determinant $\eta \otimes \mathcal{O}(-rx)$.

Remark 2.4. Let E be a vector bundle over X .

- (1) Given an element $\varphi = (x, E \xrightarrow{\rho} \mathcal{O}_x^r) = (x, E, W \subset E_x)$ in $\mathbb{G}(n-r, E)$, the r -elementary transformation of E in (x, W) is just the kernel $\text{Ker}(\rho)$ of ρ , i.e. $E^W = \text{Ker}(\rho)$.
- (2) Denote by $\text{Ker}(\iota_x)$ the kernel of the homomorphism $E_x^W \xrightarrow{\iota_x} E_x$ between the fibres at x , induced by the sheaf map ι in (2.5). It has dimension r and its annihilator $\text{Ker}(\iota_x)^\perp$ is a $n-r$ dimensional subspace in $(E_x^W)^*$, which is canonically isomorphic to W^* .
- (3) The restriction of (2.5) to x gives the exact sequence

$$(2.6) \quad 0 \rightarrow \text{Ker}(\iota_x) \rightarrow E_x^W \xrightarrow{\iota_x} E_x \xrightarrow{\alpha_W} (E_x/W) \rightarrow 0,$$

which splits as

$$(2.7) \quad 0 \rightarrow \text{Ker}(\iota_x) \rightarrow E_x^W \xrightarrow{\iota_x} W \rightarrow 0 \quad \text{and} \quad 0 \rightarrow W \rightarrow E_x \xrightarrow{\alpha_W} (E_x/W) \rightarrow 0.$$

(The extension (2.6) and its splitting will be relevant in §4, (see Proposition 4.2)).

- (4) A point $\varphi = (x, E, W \subset E_x)$ in $\mathbb{G}(n-r, E_x)$ defines the element

$$\tilde{\varphi} := (x, E^W, \text{Ker}(\iota_x) \subset E_x^W)$$

in $\mathbb{G}(r, E_x^W)$.

- (5) The r -elementary transformation of $(E^W)^*$ in (x, W^*) is E^* . That is,

$$((E^W)^*)^{W^*} = E^*,$$

and we have the exact sequence

$$(2.8) \quad \xi_{x,W^*} : 0 \rightarrow E^* \xrightarrow{\iota} (E^W)^* \rightarrow \mathcal{O}_x \times ((E^W)_x^*/W^*) \rightarrow 0.$$

We are interested in describing the set

$$\Omega := \{(x, E \xrightarrow{\rho} \mathcal{O}_x^r) : E \text{ and } \text{Ker}(\rho) \text{ are stable}\}.$$

For the stability we use the (k, ℓ) -stability.

2.3. (k, ℓ) -stability. Let k and ℓ be integers. A vector bundle E over X is (k, ℓ) -stable (see [NR2]) if for every proper subbundle F of E

$$\frac{k(n - n_F) + \ell n_F}{nn_F} < \mu(E) - \mu(F).$$

In particular, if k and ℓ are non-negative numbers, (k, ℓ) -stability implies stability. However, for negative values of k and ℓ , a (k, ℓ) -stable bundle need not be stable (see [M1]).

Denote by $A_{k,\ell}(n, d)$ the set of (k, ℓ) -stable vector bundles of rank n and degree d over X and let $A_{k,\ell}(n, \xi) := A_{k,\ell}(n, d) \cap M(n, \xi)$. The (k, ℓ) -stability is an open condition (see [NR2, Proposition 5.3]) and from [M1, Proposition 2.4] we have

Proposition 2.5. *If*

$$0 \leq k(n - 1) + \ell < (n - 1)(g - 1)$$

and

$$0 \leq k + \ell(n - 1) < (n - 1)(g - 1)$$

then $\emptyset \neq A_{k,\ell}(n, d)$ and $A_{k,\ell}(n, d) \subset M(n, d)$. Moreover $A_{k,\ell}(n, \xi)$ is non-empty.

There are natural filtration (see [M1])

$$A_{k,\ell}(n, d) \supset A_{k,\ell+1}(n, d) \supset A_{k,\ell+2}(n, d) \dots$$

and

$$A_{k,\ell}(n, d) \supset A_{k+1,\ell}(n, d) \supset A_{k+2,\ell}(n, d) \dots$$

with $A_{0,0}(n, d) = M(n, d)$ and $A_{k_0,\ell_0}(n, d) = \emptyset$ if

$$k_0(n - 1) + \ell_0 \geq (n - 1)g \quad \text{or} \quad k_0 + \ell_0(n - 1) \geq (n - 1)g.$$

In particular, (k, ℓ) -stable bundles are very general in $M(n, \xi)$. By very general we mean a point outside of a countable union of subvarieties of dimension strictly smaller than the dimension of $M(n, \xi)$.

Let E^W be the r -elementary transformation of E in $z = (x, W)$. From [NR2, Lemma 5.5] (see also [M1]) if E is (k, ℓ) -stable then E^W is $(k, \ell - r)$ -stable. Hence, we have

Proposition 2.6. *If*

(2.9)

$$0 < k(n - 1) + \ell + r < (n - 1)(g - 1) \quad \text{and} \quad 0 < k + (\ell + r)(n - 1) < (n - 1)(g - 1)$$

then $A_{k,\ell}(n, d) \neq \emptyset$ and any r -elementary transformation of a (k, ℓ) -stable bundle is $(k, \ell - r)$ -stable; in particular, stable.

Therefore, for any $z = (x, E) \in X \times A_{k,\ell}(n, d)$ we have a map

$$\phi_z : \mathbb{G}(n - r, E_x) \rightarrow M(n, d - r),$$

defined as $(x, W) \mapsto E^W$. Our aim is to define morphisms from Grassmannians to $M(n, \xi)$, for a fix $\xi \in \text{Pic}^d(X)$.

Remark 2.7. The map $\phi_z : \mathbb{G}(n - r, E_x) \rightarrow M(n, d - r)$ can be defined by considering just $(0, r)$ -stable bundles. However, for our purpose (see §5) it will be convenient to use $(k, \ell + r)$ -stable bundles that satisfies (2.9), since in that case the vector bundles in the image $\phi_E(\mathbb{G}(n - r, E))$ will be general.

3. THE MORPHISMS ϕ_z AND Υ .

We assume throughout the rest of the article, unless otherwise stated, that k, ℓ and r are integers satisfying the inequalities (2.9). For the construction of the morphisms we also assume that $(n, d) = 1$, and hence $(n, d + r) = 1$. However, we can always work in an étale cover (see [NR1, Proposition 2.4]).

To construct algebraic morphisms from Grassmannians to $M(n, \xi)$ first we define the set \mathcal{A} , which will depend on k, ℓ, r, n and ξ , but to streamline the notation, we omit such indexes.

Fix $\xi \in \text{Pic}^d(X)$. Define

$$\vartheta : X \times A_{k,\ell}(n, d + r) \rightarrow \text{Pic}^d(X) \quad \text{as} \quad (x, E) \mapsto \mathcal{O}_X(-rx) \otimes \det(E),$$

and let \mathcal{A} be the inverse image $\vartheta^{-1}(\xi)$. The natural map $\pi : \mathcal{A} \rightarrow X$ is a fibration with fibre $A_{k,\ell}(n, \xi(rx))$ at $x \in X$. The reason of considering \mathcal{A} is because the r -elementary transformation associated to the elements in \mathcal{A} , will have determinant ξ . For any $z = (x, E) \in \mathcal{A}$ we denote $\mathbb{G}(n - r, E_x)$ as $\mathbb{G}(z)$.

To prove that given $(x, E) = z \in \mathcal{A}$ the map

$$\phi_z : \mathbb{G}(z) \rightarrow M(n, \xi), \quad W \mapsto E^W,$$

is algebraic one constructs a family of stable bundles parameterised by $\mathbb{G}(z)$. The construction of the family is similar to that in [T]; [NR1]; and [NR2]. Since it will be relevant for our work, we recall the main details.

The construction goes as follows. Given $z = (x, E)$ denote by α_z the surjective homomorphism $\alpha_z : p_1^*E \rightarrow p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}$ associated to the surjective morphism $\alpha_{E_x} : \mathcal{O}_G \times E_x \rightarrow Q_{E_x}$ under the isomorphisms

$$H^0(\mathbb{G}(z), \text{Hom}(\mathcal{O}_G \times E_x, Q_{E_x})) \cong H^0(X \times \mathbb{G}(z), \text{Hom}(p_1^*E, p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x})),$$

where p_i is the projection, for $i = 1, 2$.

Since $p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}$ has a locally free resolution of length 1, the kernel of $\alpha_z : p_1^*E \rightarrow p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}$, denoted by \mathcal{P}_z , is locally free and fits in the exact sequence

$$(3.1) \quad \xi_{x,E} : 0 \rightarrow \mathcal{P}_z \rightarrow p_1^*E \xrightarrow{\alpha_z} p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x} \rightarrow 0$$

of sheaves over $X \times \mathbb{G}(z)$. The restriction of (3.1) to $X \times \{W\}$ for any $W \in \mathbb{G}(z)$ is precisely the extension (2.5), i.e. $\mathcal{P}_{z|_{X \times \{W\}}} = E^W$. Therefore, if $(x, E) = z \in \mathcal{A}$,

$$(3.2) \quad \mathcal{P}_z \rightarrow X \times \mathbb{G}(z)$$

is a family of stable bundles of rank n and determinant ξ parameterised by $\mathbb{G}(z)$ and hence we have a morphism

$$\phi_z : \mathbb{G}(z) \rightarrow M(n, \xi)$$

with the following properties.

Proposition 3.1. *Let k, ℓ and r be as in (2.9) and $(x, E) = z \in \mathcal{A}$. If r is less than the gonality of X then the morphism $\phi_z : \mathbb{G}(z) \rightarrow M(n, \xi)$ is a closed embedding for any $z \in \mathcal{A}$. Moreover, $\phi_z(\mathbb{G}(z)) \subset A_{k, \ell-r}(n, \xi)$.*

Proof. Let us first prove the injectivity of ϕ_z . Suppose, contrary to our claim, that there exist $W \neq V \in \mathbb{G}(z)$ such that $F := E^W = E^V$. Hence, $\bigwedge^n F = \xi$ where $\xi = \bigwedge^n E \otimes \mathcal{O}_X(-rx)$. From (2.5), we have two non-zero linearly independent homomorphisms $f_1, f_2 : F \rightarrow E$, that depend on W and V respectively. Choose $y \in X$ with $y \neq x$ and a linear combination $\lambda_1(f_1)_y + \lambda_2(f_2)_y$ that is not an isomorphism. Then the homomorphism $g := \lambda_1(f_1) + \lambda_2(f_2) : F \rightarrow E$ has not maximal rank on y , which is impossible. Indeed, since r is less than the gonality of X , $h^0(\mathcal{O}(rx)) \leq 1$. Thus, any morphism $\xi \rightarrow \xi \otimes \mathcal{O}(rx)$ vanish, with multiplicity r , only on x . However, the induced map $\bigwedge g^n : \bigwedge^n F \rightarrow \bigwedge^n E$ is non-zero and vanish in y and x , a contradiction. Therefore ϕ_z is injective, which is our claim.

The proof above gives more, namely

$$(3.3) \quad h^0(X, \text{Hom}(E^W, E)) = 1,$$

for any $W \in \mathbb{G}(z)$.

To prove the injectivity of the differential map

$$d\phi_z : T_{[W]}\mathbb{G}(z) \rightarrow T_{E^W}M(n, \xi)$$

recall that

$$T_{[W]}\mathbb{G}(z) = W^* \otimes E_x/W \subset (E^W)_x^* \otimes E_x/W \text{ and } T_{E^W}M(n, \xi) = H^1(X, \text{ad}(E^W)).$$

Tensor the exact sequence (2.5) with $(E^W)^*$ and apply cohomology to get the exact sequence

$$(3.4) \quad 0 \rightarrow H^0(\text{End}(E^W)) \rightarrow H^0((E^W)^* \otimes E) \rightarrow (E^W)_x^* \otimes E_x/W \xrightarrow{\delta} H^1(\text{End}(E^W)) \rightarrow \dots$$

The coboundary map $(E^W)_x^* \otimes E_x/W \xrightarrow{\delta} H^1(X, \text{End}(E^W))$ is injective because

$$H^0(X, \text{End}(E^W)) \cong H^0(X, E \otimes (E^W)^*),$$

since E^W is stable and $H^0(X, \text{Hom}(E^W, E)) \cong \mathbb{C}$ (see (3.3)).

Hence, the restriction of δ (or $(-\delta)$) to $W^* \otimes E_x/W$ is precisely the differential $d\phi_z$ (see [NR2, Lemma 5.10]). Therefore $\phi_z : \mathbb{G}(z) \rightarrow M(n, \xi)$ is a closed embedding, which proves the proposition. \square

Given a pair $(x, E) = z \in \mathcal{A}$, the image $\phi_z(\mathbb{G}(z)) \subset M(n, \xi)$ defines a closed subscheme in $M(n, \xi)$, called *r-Hecke cycle*, and hence a point $[\phi_z(\mathbb{G}(z))]$ in the Hilbert scheme Hilb_M . Since ϕ_z is an embedding, when there is no confusion, we will identify $\mathbb{G}(z)$ with its image $\phi_z(\mathbb{G}(z))$ in $M(n, \xi)$. The morphism ϕ_z is called *r-Hecke morphism* and the vector bundle $\mathcal{P}_z \rightarrow X \times \mathbb{G}(z)$ an *r-Hecke bundle*. In section §5 we will parametrise such morphisms and bundles.

The construction of the morphism ϕ_z can be done for families of (k, ℓ) -stable bundles. Indeed, let $f : \mathcal{E} \rightarrow X \times T$ be a family of (k, ℓ) -stable bundles of rank n and degree $d+r$ parameterised by T . Define $\vartheta : X \times T \rightarrow \text{Pic}^d(X)$ as $(x, t) \mapsto \mathcal{O}_X(-rx) \otimes \det(\mathcal{E}_t)$ and let $\mathcal{A}(T)$ be the inverse image $\vartheta^{-1}(\xi)$. Let $\mathbb{G}(\mathcal{E}) \xrightarrow{\pi_1} \mathcal{A}(T)$ be the Grassmannian bundle of $(n-r)$ -planes associated to the restriction of \mathcal{E} to $\mathcal{A}(T)$. Let $\gamma : \mathbb{G}(\mathcal{E}) \rightarrow X$ be the composition

$$\mathbb{G}(\mathcal{E}) \xrightarrow{\pi_1} \mathcal{A}(T) \xrightarrow{p_1} X$$

and $\Gamma := \Gamma_\gamma \subset X \times \mathbb{G}(\mathcal{E})$ the divisor associated to the graph of γ (strictly speaking, Γ_γ is in $\mathbb{G}(\mathcal{E}) \times X$). As before, over $X \times \mathbb{G}(\mathcal{E})$ we have a surjection

$$p_2^* \pi_1^*(\mathcal{E}) \xrightarrow{\tilde{\alpha}} \mathcal{O}_\Gamma \otimes p_2^* Q_\mathcal{E} \rightarrow 0,$$

where $Q_\mathcal{E}$ is the tautological quotient bundle and $p_2 : X \times \mathbb{G}(\mathcal{E}) \rightarrow \mathbb{G}(\mathcal{E})$ the projection. The kernel $\mathcal{P}_\mathcal{E}$, which fits in the exact sequence

$$(3.5) \quad 0 \rightarrow \mathcal{P}_\mathcal{E} \rightarrow p_2^* \pi_1^*(\mathcal{E}) \xrightarrow{\tilde{\alpha}} \mathcal{O}_\Gamma \otimes p_2^* Q_\mathcal{E} \rightarrow 0,$$

is locally free. Hence, $\mathcal{P}_\mathcal{E}$ is a family of stable bundles parameterised by $\mathbb{G}(\mathcal{E})$. Therefore we have a morphism

$$(3.6) \quad \Phi_\mathcal{E} : \mathbb{G}(\mathcal{E}) \rightarrow M(n, \xi)$$

and a diagram

$$(3.7) \quad \begin{array}{ccc} & \mathbb{G}(\mathcal{E}) & \\ \pi_1 \swarrow & & \searrow \Phi_\mathcal{E} \\ \mathcal{A}(T) & & M(n, \xi). \end{array}$$

In particular, applying the above construction to the family defined by the restriction of the Poincaré bundle \mathcal{U} to $X \times \mathcal{A}_{k,\ell}(n, d+r)$, we obtain the exact sequence (see (3.5))

$$(3.8) \quad 0 \rightarrow \mathcal{P}_\mathcal{U} \rightarrow p_2^* \pi_1^*(\mathcal{U}) \rightarrow \mathcal{O}_\Gamma \otimes p_2^* Q_\mathcal{U} \rightarrow 0,$$

over $X \times \mathbb{G}(\mathcal{U})$ with $\pi_1 : \mathbb{G}(\mathcal{U}) \rightarrow \mathcal{A}$; and the diagram

$$(3.9) \quad \begin{array}{ccc} & \mathbb{G}(\mathcal{U}) & \\ \pi_1 \swarrow & & \searrow \Phi \\ \mathcal{A} & & M(n, \xi). \end{array}$$

We denote $\Phi_{\mathcal{U}}$ just by Φ .

The above construction is functorial. If we denote by \mathcal{HG} the irreducible component of $Hilb_M$ that contains the r -Hecke cycles $[\phi_z(\mathbb{G}(z))]$, we get an algebraic morphism

$$(3.10) \quad \Upsilon : \mathcal{A} \rightarrow \mathcal{HG},$$

defined as $(x, E) \mapsto [\phi_z(\mathbb{G}(z))]$.

Proposition 3.2. *If $(n, d) = 1$ and r is less than the gonality of X then $\Upsilon : \mathcal{A} \rightarrow \mathcal{HG}$ is injective.*

Proof. Suppose the proposition were false. Then we could find $(x, E) = z_1 \neq z_2 = (y, F)$ in \mathcal{A} such that $[\phi_{z_1}(\mathbb{G}(z_1))] = [\phi_{z_2}(\mathbb{G}(z_2))]$. Since ϕ_{z_1} and ϕ_{z_2} are embeddings, we get an isomorphism $\beta : \mathbb{G}(z_1) \rightarrow \mathbb{G}(z_2)$ that induces the following commutative diagrams

$$\begin{array}{ccc} \mathbb{G}(z_1) & & \\ \downarrow \beta & \searrow \phi_{z_1} & \\ & & M(n, \xi) \\ & \nearrow \phi_{z_2} & \\ \mathbb{G}(z_2) & & \end{array}$$

and

$$\begin{array}{ccc} X \times \mathbb{G}(z_1) & \xrightarrow{(id, \beta)} & X \times \mathbb{G}(z_2) \\ & \searrow p_1 & \nearrow \tilde{p}_1 \\ & & X \end{array}$$

i.e. $\phi_{z_1} = \phi_{z_2} \circ \beta$ and $\tilde{p}_1 \circ (id, \beta) = p_1$.

By the universal properties of $M(n, \xi)$,

$$(3.11) \quad \mathcal{P}_{z_1} \cong (id, \beta)^*(\mathcal{P}_{z_2}) \otimes p_2^*(L)$$

where L a line bundle on $\mathbb{G}(z_1)$. But L is trivial since $(\mathcal{P}_{z_1})|_{\{t\} \times \mathbb{G}(z_1)}$ is trivial, for any $t \neq x$ (see (3.1)). We thus get $\mathcal{P}_{z_1} \cong (id, \beta)^*(\mathcal{P}_{z_2})$ and $p_{1*}\mathcal{P}_{z_1} = \tilde{p}_{1*}\mathcal{P}_{z_2}$.

Let

$$0 \rightarrow p_{1*}\mathcal{P}_{z_1} \rightarrow p_{1*}(p_1^*E) \xrightarrow{\alpha_*} p_{1*}(p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}) \rightarrow \dots$$

be the direct image sequence of (3.1) by p_1 . It follows that $p_{1*}(\mathcal{P}_{z_1}) \cong E(-x)$, because

- (1) $E \cong p_{1*}p_1^*(E)$,
- (2) $p_{1*}(p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}) \cong \mathcal{O}_x \otimes E$,
- (3) the morphism α_* is the morphism associate to α_{E_x} ,
- (4) the kernel of $E \xrightarrow{\alpha_*} \mathcal{O}_x \otimes E$ is $E(-x)$.

Similarly, $\tilde{p}_{1*}(\mathcal{P}_{z_2}) \cong F(-y)$. Therefore, since r is less than the gonality of X , the isomorphisms

$$\begin{aligned}
 E(-x) &\cong p_{1*}(\mathcal{P}_{z_1}) \\
 &\cong p_{1*}(id, \beta)^*(\mathcal{P}_{z_2}) \\
 &\cong \tilde{p}_{1*}(\mathcal{P}_{z_2}) \\
 &\cong F(-y)
 \end{aligned}$$

imply that $x = y$ and $E = F$, which proves the proposition. \square

4. THE HILBERT SCHEME.

To compute the differential map $d\Upsilon$ at $z = (x, E) \in \mathcal{A}$ we denote by $N_{\mathbb{G}/M}$ the normal bundle of $\mathbb{G}(z)$ in $M(n, \xi)$ and by TM the tangent bundle of $M(n, \xi)$. These bundles fit into the following the exact sequence

$$(4.1) \quad 0 \rightarrow T\mathbb{G}(z) \rightarrow \phi_z^*TM \rightarrow N_{\mathbb{G}/M} \rightarrow 0$$

of vector bundles over $\mathbb{G}(z)$. Then Theorem 1.1 follows from the next proposition.

Proposition 4.1. *For any $(x, E) = z \in \mathcal{A}$,*

- (1) $H^0(\mathbb{G}(z), N_{\mathbb{G}/M}) = T_z\mathcal{A}$.
- (2) $H^i(\mathbb{G}(z), N_{\mathbb{G}/M}) = 0$ for $i \geq 1$.
- (3) $N_{\mathbb{G}/M}$ is generated by global sections.

The proof of Proposition 4.1 is somewhat lengthy, so we will split it into several lemmas.

From diagram (3.9) we have that for any $(x, E) = z \in \mathcal{A}$, $\pi_1^{-1}(z) = \mathbb{G}(z)$ and $\Phi|_{\mathbb{G}(z)} = \phi_z$. Hence, also from (3.9) we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T_{\Phi|_{\mathbb{G}(z)}} & = & T_{\Phi|_{\mathbb{G}(z)}} & & \\
 & & \downarrow & & \downarrow & & \\
 (4.2) \quad 0 & \rightarrow & T\mathbb{G}(z) & \rightarrow & T\mathbb{G}(\mathcal{U})|_{\mathbb{G}(z)} & \rightarrow & N_{\mathbb{G}/\mathbb{G}(\mathcal{U})} \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & T\mathbb{G}(z) & \rightarrow & \phi_z^*TM & \rightarrow & N_{\mathbb{G}/M} \rightarrow 0, \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where $N_{\mathbb{G}/\mathbb{G}(\mathcal{U})}$ is the normal bundle of $\mathbb{G}(z)$ in $\mathbb{G}(\mathcal{U})$.

Since $\mathbb{G}(z)$ is a fibre of the Grassmannian bundle $\pi_1 : \mathbb{G}(\mathcal{U}) \rightarrow \mathcal{A}$, the normal bundle $N_{\mathbb{G}/\mathbb{G}(\mathcal{U})}$ is trivial, i.e. $N_{\mathbb{G}/\mathbb{G}(\mathcal{U})} = \mathcal{O}_{\mathbb{G}(z)} \times T_z \mathcal{A}$. Hence, we have the exact sequence

$$(4.3) \quad 0 \rightarrow T_{\Phi|_{\mathbb{G}(z)}} \rightarrow \mathcal{O}_{\mathbb{G}(z)} \times T_z \mathcal{A} \rightarrow N_{\mathbb{G}/M} \rightarrow 0.$$

In order to compute the cohomology of $T_{\Phi|_{\mathbb{G}(z)}}$, we restrict the extension (3.8) to the divisor $\Gamma_\gamma = \mathbb{G}(\mathcal{U})$. To streamline the notation, we use the same notation for the restrictions.

The next proposition is slightly different from [NR1, Lemma 2.1, Proposition 4.8 and 4.12], but the proof works verbatim (see also [T] and Remark 2.4).

Proposition 4.2. *The restriction of (3.8) to the divisor Γ_γ induces the exact sequence*

$$(4.4) \quad 0 \rightarrow \mathcal{K} \rightarrow \mathcal{P}_{\mathcal{U}} \rightarrow p_2^* \pi_1^* \mathcal{U} \xrightarrow{\alpha} Q_{\mathcal{U}} \rightarrow 0$$

of vector bundles over $\Gamma_\gamma = \mathbb{G}(\mathcal{U})$, where \mathcal{K} is a vector bundle of rank r such that $\mathcal{K} \cong Q_{\mathcal{U}}$. Moreover, the sequence (4.4) splits as

$$(4.5) \quad 0 \rightarrow \mathcal{K} \rightarrow \mathcal{P}_{\mathcal{U}} \rightarrow S_{\mathcal{U}} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow S_{\mathcal{U}} \rightarrow p_2^* \pi_1^* \mathcal{U} \xrightarrow{\alpha} Q_{\mathcal{U}} \rightarrow 0.$$

Note that the restriction of \mathcal{K} and of the sequences (4.5) to $(x, E, W \subset E_x) \in \mathbb{G}(\mathcal{U})$ is the vector space $\text{Ker}(\iota_x)$ and the sequences (2.7) in Remark 2.4.

Lemma 4.3. *The Grassmannian bundle $g : \mathbb{G}(r, \mathcal{P}_{\mathcal{U}}) \rightarrow \mathbb{G}(\mathcal{U})$ has a section*

$$\sigma_{\mathcal{K}} : \mathbb{G}(\mathcal{U}) \rightarrow \mathbb{G}(r, \mathcal{P}_{\mathcal{U}})$$

such that

- (1) $\sigma_{\mathcal{K}}^*(T_g)_{|\mathbb{G}(z)} \cong \Omega^1 \mathbb{G}(z)$.
- (2) $\sigma_{\mathcal{K}}^*(T_{\pi_2|_{\sigma_{\mathcal{K}}(\mathbb{G}(z))}}) \cong (T_{\Phi})_{|\mathbb{G}(z)}$, where $\pi_2 : \mathbb{G}(r, \mathcal{P}_{\mathcal{U}}) \rightarrow M(n, \xi)$ is defined as $(x, F, Z \subset F_x) \mapsto F$.
- (3) $H^i(\mathbb{G}(z), (T_{\Phi})_{|\mathbb{G}(z)}) \cong H^i(\mathbb{G}(z), \sigma_{\mathcal{K}}^*(T_{\pi_2|_{\sigma_{\mathcal{K}}(\mathbb{G}(z))}}))$ for $i \geq 0$.

Proof. The vector bundle \mathcal{K} has rank r , and defines the section $\sigma_{\mathcal{K}} : \mathbb{G}(\mathcal{U}) \rightarrow \mathbb{G}(r, \mathcal{P}_{\mathcal{U}})$ such that $\sigma_{\mathcal{K}}^*(T_g) \cong \mathcal{K}^* \otimes \mathcal{P}_{\mathcal{U}}/\mathcal{K}$. Thus, from Proposition 4.2 we have that $\sigma_{\mathcal{K}}^*(T_g) \cong Q_{\mathcal{U}}^* \otimes S_{\mathcal{U}} \cong \Omega_{\pi_1}^1$.

The rest of the lemma follow from the definition of $\sigma_{\mathcal{K}}$, since it is clear that $\pi_2 \circ \sigma_{\mathcal{K}} = \Phi$, and that $\sigma_{\mathcal{K}}$ is an isomorphism with its image. Therefore, $T_{\Phi|_{\mathbb{G}(z)}} \cong \sigma_{\mathcal{K}}^*(T_{\pi_2|_{\sigma_{\mathcal{K}}(\mathbb{G}(z))}})$ as claimed. \square

The situation is summed up in, and we hope clarified by, the following commutative diagram

$$(4.6) \quad \begin{array}{ccccc} & \mathbb{G}(\mathcal{U}) & \xrightleftharpoons[\substack{g \\ \sigma_{\mathcal{K}}}]{} & \mathbb{G}(r, \mathcal{P}_{\mathcal{U}}) \subset \mathbb{G}(r, \mathcal{P}) & \\ \pi_1 \swarrow & & \searrow \Phi & \downarrow \pi_2 & \searrow \pi \\ & \mathcal{A} & & M(n, \xi) & X \times M(n, \xi), \\ & & & \nwarrow p_2 & \end{array}$$

where $\pi : \mathbb{G}(r, \mathcal{P}) \rightarrow X \times M(n, \xi)$ is the Grassmannian Poincaré bundle.

Remark 4.4. Strictly speaking, the morphisms in the above diagram are defined in open sets. Let $\widetilde{\mathbb{G}\mathcal{U}} := \Phi^{-1}(\Phi(\mathbb{G}(n-r, \mathcal{U})))$ and $\mathcal{A}^s := \pi_1(\widetilde{\mathbb{G}\mathcal{U}})$. The open sets, and mainly their codimension, are relevant to compute cohomology groups of $M(n, \xi)$ (see [NR1]). However, we will not use this fact in any essential way, so we use $\mathbb{G}(\mathcal{U}), \mathbb{G}(r, \mathcal{P}_{\mathcal{U}}), M(n, \xi)$ as target of our morphisms.

Lemma 4.5. $H^0(\sigma_{\mathcal{K}}(\mathbb{G}(z)), (T_g)_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))}) \cong H^0(\sigma_{\mathcal{K}}(\mathbb{G}(z)), (T_{\pi_2})_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))})$. Moreover, $H^i(\sigma_{\mathcal{K}}(\mathbb{G}(z)), (T_{\pi_2})_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))}) = 0$ for $i \geq 1$.

Proof. The fibre of π_2 at $E^W \in \phi_z(\mathbb{G}(n-r, E_x))$ is the Grassmannian bundle

$$p : \mathbb{G}(r, E^W) \rightarrow X,$$

and hence the tangent in the fibre $\pi_2^{-1}(E^W)$ fits in the exact sequence

$$0 \rightarrow T_p \rightarrow T\mathbb{G}(r, E^W) \rightarrow p^*TX \rightarrow 0.$$

Since ϕ_z is an embedding, the image $\sigma_{\mathcal{K}}(\mathbb{G}(z))$ is transversal to the fibres of π_2 . Therefore, $(T_{\pi_2})_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))}$ fits into the following exact sequence

$$(4.7) \quad \zeta : 0 \rightarrow (T_g)_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))} \rightarrow (T_{\pi_2})_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))} \rightarrow \mathcal{O}_{\sigma_{\mathcal{K}}(\mathbb{G}(z))} \times T_x X \rightarrow 0.$$

Hence, we have the first part of the lemma by recalling that $\sigma_{\mathcal{K}}(\mathbb{G}(z)) \subset \mathbb{G}(r, \mathcal{P}_x)$ and from the cohomology of the sequence (4.7). From this it follows that

$$\begin{aligned} H^0(\sigma_{\mathcal{K}}(\mathbb{G}(z)), (T_{\pi_2})_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))}) &= 0, \text{ since, by Lemma 4.3, (1)} \\ H^0(\sigma_{\mathcal{K}}(\mathbb{G}(z)), (T_g)_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))}) &\cong H^0(\mathbb{G}(z), \Omega^1 \mathbb{G}(z)) = 0. \end{aligned}$$

From the cohomology of (4.7) and Remark 2.1 it follows that $H^i(\sigma_{\mathcal{K}}(\mathbb{G}(z)), (T_{\pi_2})_{|\sigma_{\mathcal{K}}(\mathbb{G}(z))}) = 0$ for $i \geq 1$, which completes the proof. \square

Proof of Proposition 4.1 We first recall (see (4.3)) that for any $(x, E) = z \in \mathcal{A}$, the normal bundle $N_{\mathbb{G}/M}$ fits in the exact sequence

$$(4.8) \quad 0 \rightarrow T_{\Phi|_{\mathbb{G}(z)}} \rightarrow \mathcal{O}_{\mathbb{G}(z)} \times T_z \mathcal{A} \rightarrow N_{\mathbb{G}/M} \rightarrow 0.$$

We claim that $H^i(\mathbb{G}(z), T_{\Phi|_{\mathbb{G}(z)}}) = 0$ for $i \geq 0$. Indeed, from Lemmas 4.3,(3) and 4.5 $H^i(\mathbb{G}(z), T_{\Phi|_{\mathbb{G}(z)}}) = 0$ for $i \geq 0$.

Consequently, $H^0(\mathbb{G}(z), N_{\mathbb{G}/M}) = T_z\mathcal{A}$ and $N_{\mathbb{G}/M}$ is generated by global sections. Moreover, $H^i(\mathbb{G}(z), N_{\mathbb{G}/M}) = 0$ for $i \geq 1$, which is the desired conclusion. \square

The next proposition will be used to fix the Hilbert polynomial.

Proposition 4.6. *For any $(x, E) = z \in \mathcal{A}$, $\deg(\phi_z) = 2n$.*

Proof. From the exact sequences (4.3), (4.7), diagram (4.2) and Lemma 4.3, we have that $\deg N_{\mathbb{G}/M} = n$. Hence, from (4.1), $\deg(\phi_z^*(TM)) = 2n$. Therefore, $\deg(\phi_z) = \deg(\phi_z^*(-K_M)) = 2n$. \square

Let P be the Hilbert polynomial

$$P(m) = \chi(\mathbb{G}, \phi_z^*(-K_M)) = \chi(\mathbb{G}, m(\mathcal{O}_{\mathbb{G}}(2n))).$$

Theorem 4.7. *If $(n, d) = 1$ and r is less than the gonality of X then*

- (1) *there is an algebraic isomorphism Υ from \mathcal{A} to an open subscheme of \mathcal{HG} .*
- (2) *The Hilbert scheme is smooth at $[\Upsilon(z)]$, for any $z \in \mathcal{A}$.*
- (3) $\dim \mathcal{HG} = (n^2 - 1)(g - 1) + 1$.
- (4) *The deformations of r -Hecke cycles are r -Hecke cycles.*

Proof. From Proposition 3.2 the morphism $\Upsilon : \mathcal{A} \rightarrow \mathcal{HG}$ is injective and clearly the isomorphism $T_z\mathcal{A} \rightarrow H^0(\mathbb{G}(z), N_{\mathbb{G}/M})$ in Proposition 4.1 is the differential of Υ at z .

Since $H^i(\mathbb{G}(z), N_{\mathbb{G}/M}) = 0$ for $i \geq 1$, \mathcal{HG} is smooth at $[\phi_z(\mathbb{G}(z))]$ for all $z \in \mathcal{A}$ and $\dim \mathcal{HG} = (n^2 - 1)(g - 1) + 1$. Moreover, from the exact sequence (4.3), $N_{\mathbb{G}/M}$ is generated by global sections and the deformations of an r -Hecke cycle are r -Hecke cycles. \square

5. THE $Mor(\mathbb{G}, M(n, \xi))$ SCHEME.

Let \mathbb{G} be the Grassmannian $\mathbb{G}(n-r, \mathbb{C}^n)$. In this section we apply the previous results to describe an open set of a component of the scheme $Mor_P(\mathbb{G}, M(n, \xi))$ and $Mor_P(\mathbb{P}^1, M(n, \xi))$, where P is the Hilbert polynomial

$$P(m) = \chi(\mathbb{G}, m(\mathcal{O}_{\mathbb{G}}(2n))).$$

Let $Hom_P(\mathbb{G}, M(n, \xi))$ be the scheme of morphisms from \mathbb{G} to $M(n, \xi)$. Recall that to remove the dependency on the choice of coordinates of \mathbb{G} , we take the quotient by the action of $Aut(\mathbb{G})$. Therefore,

$$Mor_P(\mathbb{G}, M(n, \xi)) = Hom_P(\mathbb{G}, M(n, \xi))/Aut(\mathbb{G}),$$

which can be defined by means of the Chow variety (see [G], [BDW] and [PT]).

Proposition 5.1. *Any embedding $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ passing through general points has degree at least $2n$, with respect to $-K_M$.*

Proof. Suppose that $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ has degree t . Let $\kappa : \mathbb{P}^1 \rightarrow \mathbb{G}$ be a morphism of degree 1. Since $\lambda : \mathbb{G} \rightarrow M(n, \xi)$ is an embedding, the composition $\lambda \circ \kappa : \mathbb{P}^1 \rightarrow M(n, \xi)$ is a rational curve and, from [S, Theorem 1], it has degree $\deg(\lambda \circ \kappa) = t \geq 2n$, and the proposition follows. \square

Corollary 5.2. *If the conditions (2.9) are satisfied then the r -Hecke morphisms have minimal degree.*

Proof. The conditions (2.9) imply that the vector bundles in the image $\phi_z(\mathbb{G}(z))$ are (k, ℓ) -stable and hence very general (see Remark 2.7). Propositions 5.1 and 4.6 now show that $\deg(\phi_z)$ is minimal, which is our claim. \square

The next proposition follows directly from Proposition 3.2.

Proposition 5.3. *The morphism $\Sigma : \mathcal{A} \rightarrow \text{Mor}_P(\mathbb{G}, M(n, \xi))$, defined as $z \mapsto [\phi_z]$ is algebraic and injective. Moreover, $\Sigma(\mathcal{A})$ is contained in an irreducible component of $\text{Mor}_P(\mathbb{G}, M(n, \xi))$.*

Denote by $\text{Mor}_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi))$ the irreducible component of $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ that contains the r -Hecke morphisms.

Theorem 5.4. *If $(n, d) = 1$ and r is less than the gonality of X then the scheme $\text{Mor}_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi))$ is smooth at the r -Hecke morphisms and has dimension $(n^2 - 1)(g - 1) + 1$. In particular, $\dim \text{Mor}_P^{\mathcal{H}, 2}(\mathbb{P}^2, M(3, \xi)) = 8g - 7$.*

Proof. The theorem follows from [G], since from the exact sequence (4.1) and Proposition 4.2

$$\dim H^0(\mathbb{G}(z), \phi_z^* TM) = \dim H^0(\mathbb{G}(z), T\mathbb{G}(z)) + \dim H^0(\mathbb{G}(z), N_{\mathbb{G}/M})$$

and $H^i(\mathbb{G}(z), \phi_z^*(TM)) = 0$ for all $i \geq 1$. Recall that the image of the inclusion $H^0(\mathbb{G}(z), T\mathbb{G}(z)) \rightarrow H^0(\mathbb{G}(z), \phi_z^*(TM))$ corresponds to the deformations of ϕ_z by reparameterisations. Hence, the dimension of $\text{Mor}_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi))$ is $\dim H^0(\mathbb{G}, N_{\mathbb{G}/M}) = (n^2 - 1)(g - 1) + 1$. This completes the proof of the theorem. \square

Let us mention two important consequences of the theorem.

Corollary 5.5. *If $\text{Mor}_{2n}^{\mathcal{H}}(\mathbb{P}^1, M(n, \xi))$ is the irreducible component of the space of Hecke curves in $M(n, \xi)$ then $\mathbb{G}(2, \mathcal{U}) \subseteq \text{Mor}_{2n}^{\mathcal{H}}(\mathbb{P}^1, M(n, \xi))$, where $\mathbb{G}(2, \mathcal{U})$ is the restriction of the Grassmannian Poincaré bundle to \mathcal{A} .*

Proof. The proof is based in the following observation. The Hecke curves are lines in the projective space $\mathbb{P}(E_x)$. Therefore, the corollary follows from the identification of $\mathbb{G}(1, \mathbb{P}(E_x))$ and $\mathbb{G}(2, E_x)$. \square

Recall that the moduli space $Mor_s(\mathbb{P}^1, \mathbb{G})$ of stable maps from \mathbb{P}^1 to the Grassmannian \mathbb{G} , of degree s , is a smooth quasi-projective variety of dimension $ns + r(n - r)$. It admits various natural compactifications, for example by the Hilbert scheme of the graph (see [LT]) or by the Grothendieck scheme of quotient sheaves of a trivial bundle on \mathbb{P}^1 (see [BDW]).

Corollary 5.6. *If $Mor_s(\mathbb{P}^1, \mathbb{G}) \neq \emptyset$ then $Mor_{2ns}(\mathbb{P}^1, M(n, \xi)) \neq \emptyset$. Moreover, $\dim Mor_{2ns}(\mathbb{P}^1, M(n, \xi)) \geq (n^2 - 1)(g - 1) + 1$.*

Proof. The proof is immediate from the natural morphism

$$Mor_s(\mathbb{P}^1, \mathbb{G}) \times Mor_P(\mathbb{G}, M(n, \xi)) \rightarrow Mor_{2ns}(\mathbb{P}^1, M(n, \xi)),$$

induced by the composition. \square

The remainder of this section will be devoted to the proof of Theorem 1.4.

Let us recall that if L is an ample divisor on a projective variety Y , the L -degree $\deg_L(\mathcal{E})$ of a torsion-free sheaf \mathcal{E} on Y is defined to be the intersection number $[c_1(\mathcal{E})] \cdot [L]^{\dim Y - 1}$. The torsion-free sheaf \mathcal{E} is said to be L -stable if, for every proper subsheaf \mathcal{F} of \mathcal{E} ,

$$\frac{\deg_L \mathcal{F}}{n_{\mathcal{F}}} < \frac{\deg_L \mathcal{E}}{n_{\mathcal{E}}}.$$

Denote by $M_Y^L(\mathcal{E})$ the moduli space of L -stable sheaves over Y with the same numerical invariants as \mathcal{E} .

For any $z = (x, E) \in \mathcal{A}$, the morphism ϕ_z is defined by the existence of a vector bundle \mathcal{P}_z over $X \times \mathbb{G}$ called r -Hecke bundle. Since $Pic(\mathbb{G}) = \mathbb{Z}$, $Pic(X \times \mathbb{G}) = Pic(X) \oplus Pic(\mathbb{G})$. Thus, any polarization L on $X \times \mathbb{G}$ can be expressed in the form $L = a\alpha + b\beta$ with $a, b > 0$, where α is ample on X and β ample on \mathbb{G} .

Proposition 5.7. *For any $(x, E) = z \in \mathcal{A}$, the r -Hecke bundle $\mathcal{P}_z \rightarrow X \times \mathbb{G}$ is L -stable with respect to any polarization L .*

Proof. The vector bundle $\mathcal{P}_z \rightarrow X \times \mathbb{G}$ is a family of stable bundles parameterised by \mathbb{G} . By construction, for any $y \neq x$, $\mathcal{P}_{z|_{\{y\} \times \mathbb{G}}} \cong \mathcal{O}_{\mathbb{G}}^n$. Therefore, the L -stability of \mathcal{P}_z follows from [BBN, Lemma 2.2]. \square

Let $M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$ be the irreducible component of the moduli spaces of L -stable vector bundles over $X \times \mathbb{G}$ containing the r -Hecke bundles \mathcal{P}_z .

Theorem 5.8. *If $(n, d) = 1$ and r is less than the gonality of X then*

- (1) the algebraic morphism $\Gamma : \mathcal{A} \rightarrow M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$ defined as $z \mapsto [\mathcal{P}_z]$ is injective.
- (2) $M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$ is smooth at $\Gamma(z)$, for all $z \in \mathcal{A}$.
- (3) $\dim M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z) = n^2g + 1$.
- (4) $\dim M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)/\text{Aut}(X \times \mathbb{G}) = (n^2 - 1)(g - 1) + 1$.

Proof. The tangent space of the moduli space at \mathcal{P}_z is

$$H^1(X \times \mathbb{G}, \text{End}(\mathcal{P}_z)) = H^1(X \times \mathbb{G}, \mathcal{O}_{X \times \mathbb{G}}) \oplus H^1(X \times \mathbb{G}, \text{Ad}(\mathcal{P}_z)).$$

From the Leray spectral sequences and the cohomology of (4.1) we have that

$$H^1(X \times \mathbb{G}, \text{Ad}(\mathcal{P}_z)) = H^0(\mathbb{G}, \phi_z^*(TM)) = H^0(\mathbb{G}, T\mathbb{G}(z)) \oplus H^0(\mathbb{G}, N_{\mathbb{G}/M}).$$

Since $H^1(X \times \mathbb{G}, \mathcal{O}_{X \times \mathbb{G}}) = H^1(X, \mathcal{O}_X)$,

$$H^1(X \times \mathbb{G}, \text{End}(\mathcal{P}_z)) = H^0(\mathbb{G}, T\mathbb{G}(z)) \oplus H^0(\mathbb{G}, N_{\mathbb{G}/M}) \oplus H^1(X, \mathcal{O}_X).$$

Therefore, the theorem follows from the equality $h^0(\mathbb{G}, N_{\mathbb{G}/M}) = \dim T_z \mathcal{A} = (n^2 - 1)(g - 1) + 1$. Note that locally the deformations of \mathcal{P}_z come from those of the curve, of the Grassmannian and from $H^0(\mathbb{G}, \phi_z^*(TM))$, and this is precisely the assertion of the theorem. \square

6. OPEN QUESTIONS

There are a number of interesting questions about the r -Hecke cycles, the Hecke morphisms and the Hecke bundles. We mention some of them, which are most interesting from the viewpoint of moduli spaces.

We have defined an open set:

- (1) $\Upsilon(\mathcal{A})$ of the Hilbert scheme $\mathcal{H}\mathcal{G}$,
- (2) $\Sigma(\mathcal{A})$ of the moduli scheme $Mor_P^{\mathcal{H}, r}(\mathbb{G}, M(n, \xi))$,
- (3) $\Gamma(\mathcal{A})$ of the moduli space $M_{X \times \mathbb{G}}^{L, \mathcal{H}}(\mathcal{P}_z)$.

Let $\tilde{\Upsilon}$, $\tilde{\Sigma}$ and $\tilde{\Gamma}$ be the closures of $\Upsilon(\mathcal{A})$, $\Sigma(\mathcal{A})$ and $\Gamma(\mathcal{A})$ respectively.

The natural questions are:

Question 6.1. *Which are the elements that compactify these open sets? Does the boundary points have natural geometric meaning?*

Question 6.2. *What are the relationships among $\tilde{\Upsilon}$, $\tilde{\Sigma}$ and $\tilde{\Gamma}$?*

Question 6.3. *Does they give a new compactification of the moduli space $M(n, \xi)$?*

For each $\phi_z : \mathbb{G} \rightarrow M(n, \xi)$, the image of the differential $d\phi_z$ define a subspace $d\phi_z T\mathbb{G}$ of $T_{E^W} M$. The dual of the quotient $T_{E^W} M / d\phi_z T\mathbb{G}$ defines a subspace $\mathbb{G}\mathbb{H}$ in the cotangent space

$$T_{E^W}^* M = H^0(X, \text{End} E^W \otimes K_X),$$

and hence a subspace of the Higgs bundles and the spectral curves.

Question 6.4. *Describe the Higgs bundles and the spectral curves defined by GH.*

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